MATH 135 Assignment 3

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Problem 1

Part A:

This statement is **false**. Counterexample: $n = 1$

$$3^2 - 3 - 3$$

$$= 3$$

Since 3 is odd, this statement is false. $\square$

Part B:

This statement is **true**. *Proof*: Since n is odd, it can be written as $2k + 1, k \in \mathbb{Z}$. Then,

$$n^2 - n - 3 = (2k + 1)^2 - (2k + 1) - 3$$

$$= 4k^2 + 4k + 1 - 2k - 1 - 3$$

$$= 4k^2 + 2k - 3$$

$$= 2(2k^2 + k - 2) + 1$$

Since $2k^2 + k - 2$ is an integer, $n^2 - n - 3$ can be written as $2z + 1, z \in \mathbb{Z}$. Therefore, $n^2 - n - 3$ is odd.

Part C:

This statement is **true**. *Proof*:

Case 1: n is odd: this was proved in Problem 1b.

Case 2: n is even. Then, n can be written as $2k, k \in \mathbb{Z}$.

$n^2 - n - 3 = (2k)^2 - 2k - 3$

$$= 4k^2 - 2k - 3$$

$$= 2(2k^2 - k - 2) + 1$$

Since $2k^2 - k - 2$ is an integer $n^2 - n - 3$ can be written as $2z + 1, z \in \mathbb{Z}$. Therefore, $n^2 - n - 3$ is odd. $\square$

Part D:

This statement is **true**. We will prove it using the contrapositive, which can be stated as:

If n is odd, then $n^2 - n - 3$ is odd. This was proven Problem 1b.

Part E:

This statement is **false**. Counterexample: $n = 4$

$$n^2 - n - 3 = 4^2 - 4 - 3$$

$$= 16 - 4 - 3$$

$$= 9$$

Since $n = 4$ is even, and $n^2 - n - 3 = 9$, which is odd, therefore, this statement is false.

Problem 2

Part A:

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$$

Part B:

Statement:

$$\neg(\exists k \in \mathbb{N}, \forall a \in \mathbb{N}, a^2 + 15a + k \text{ is prime})$$

$$\equiv \forall k \in \mathbb{N}, \exists a \in \mathbb{N}, a^2 + 15a + k \text{ is not prime}$$

Proof:

Case 1: For all $k \in \mathbb{N}, k \neq 1$. Choose $a = k$

$$a^2 + 15a + k$$

$$= k^2 + 15k + k$$

$$= k^2 + 16k$$

$$= k(k + 16)$$

Since $k \neq 1$ and $k + 16 > 1$, therefore, $a^2 + 15a + k$ can be expressed as a product of two natural numbers k and $k + 16$ such that neither of them are 1. Hence, $a^2 + 15a + k$ is not prime.

Case 2: $k = 1$. Choose $a = 2$. Then,

$$a^2 + 15a + k$$

$$= 2^2 + 15(2) + 1$$

$$= 35$$

Since 35 can be written as 5×7, it is not a prime number.

Therefore, for all $k \in \mathbb{N}$, there exists $a \in \mathbb{N}$ such that $a^2 + 15a + k$ is not prime. $\square$

Problem 3

Part A:

We will prove this statement using its contrapositive, which is as follows:

$$\text{If } 3a \mid (4b-1) \text{ and } a \mid (12b+7), \text{then either } a = 1 \text{ or } a = 5$$

Assuming the hypothesis,

$$4b - 1 = 3ap \tag{1}$$

and

$$12b + 7 = aq \tag{2}$$

(1) $\times 3 -$ (2) gives us:

$$10 = ar - 9ak$$

$$10 = a(r - 9k)$$

Since 10 here is equal to the product of two integers, one of which is a, therefore a must be a factor of 10. Hence, a must be one of $1, 2, 5,$ or 10.

Furthermore, equation 2 indicates that

$$12b + 7 = aq$$

$$2(6b + 3) + 1 = aq$$

Because $6b + 3 \in \mathbb{Z}$, aq can be written as $2z + 1, z \in \mathbb{Z}$, therefore aq must be odd.

Lemma 1

Lemma: If aq is odd, then a and q must both be odd.

Proof: We will prove this lemma using its contrapositive, which states that if a or q are even, then aq must be even. This is true because if a is even, then

$$aq = 2kq, k \in \mathbb{Z}$$

$$aq = 2(kq)$$

Since $kq \in \mathbb{Z}$, aq must be even.

Similarly, if q is even, aq must also be even. □

Due to Lemma 1, a must be an odd number, which rules out 2 and 10 from the possibilities of a.

Therefore, a must be either 1 or 5, as required. □

Part B:

For all positive integers a and b, if $3a \nmid (4b-1)$ or $a \nmid (12b+7)$, then $a \neq 1$ and $a \neq 5$

Part C:

The converse is **false**. Counterexample: $a = 1, b = 2$.

$$3(1) \nmid (4 \cdot 2 - 1)$$

$$3 \nmid 7 \rightarrow \text{true}$$

Therefore, the hypothesis is true since it is contains an “or” logical operator.

$$a \neq 1 \rightarrow \text{false}$$

Therefore, the conclusion is false since it contains an “and” logical operator.

Since the hypothesis failed to imply the conclusion, therefore the converse is false. □

Problem 4

Assume, for contradiction, that $\log_6 41$ is rational. Then, there exist nonnegative integers p and q such that

$$\log_6 41 = \frac{p}{q}$$

By manipulating the above statement,

$$6^{\log_6 41} = 6^{\frac{p}{q}}$$

$$41 = 6^{\frac{p}{q}}$$

$$41^q = 6^p$$

Lemma 2

Lemma: If $n \in \mathbb{Z}$ is even, then n^m is also even for $m \in \mathbb{Z}, m \geq 1$.

Proof: Since n is even, it can be expressed as $2k, k \in \mathbb{Z}$.

$$n^m = (2k)^m = 2^m k^m = 2(2^{m-1} k^m)$$

Because $2^{m-1}k^m$ is an integer, n^m can be expressed as $2z, z \in \mathbb{Z}$. Therefore, n^m is also even. □

Lemma 3

Lemma: If $n \in \mathbb{Z}$ is odd, then n^m is also odd for $m \in \mathbb{Z}, m \geq 1$.

Proof by induction: Base case: $m = 1$. $n^m = n$ which is also odd.

Inductive step: assuming n^m is odd, we wish to show that n^{m+1} is also odd.

$$n^m = 2p + 1, p \in \mathbb{Z}$$

$$n^m \cdot m = (2p + 1)(2q + 1), q \in \mathbb{Z}$$

$$n^{m+1} = 2pq + 2p + 2q + 1$$

$$n^{m+1} = 2(pq + p + q) + 1$$

Since $pq + p + q \in \mathbb{Z}$, n^{m+1} is also odd. Therefore, by the principle of induction, n^m is always odd for $m \in \mathbb{Z}, m \geq 1$. □

By Lemma 3, 41^q is odd since 41 is odd, and q cannot be 0 due to it being on the denominator.

By Lemma 2, 6^p is even since 6 is odd, and p cannot be 0 since that would make $\log_6 41 = 0$ which is false.

Since an odd number cannot equal an even number, this creates a contradiction. Therefore, $\log_6 41$ must be irrational. □

Problem 5

First, we will prove that $x^2 + 2xy - 3y^2 < 0$ implies

$(y > 0 \text{ and } y > x > -3y)$ or $(y < 0 \text{ and} - 3y > x > y)$:

$$x^2 + 2xy - 3y^2 < 0$$

$$x^2 + 2xy + y^2 - 4y^2 < 0$$

$$(x + y)^2 - 4y^2 < 0$$

$$(x + y)^2 - (2y)^2 < 0$$

$$(x + y + 2y)(x + y - 2y) < 0$$

$$(x + 3y)(x - y) < 0$$

In order for the product of two real numbers to be negative, exactly one of those numbers is negative. Therefore, we have two cases:

Case 1:

$$x + 3y > 0 \text{ and } x - y < 0$$

$$y > x$$

$$x > -3y$$

$$y > x > -3y$$

$$y > -3y$$

Case 1a: $y > 0$. $1 > -3$, which is true for all y. Case 1b: $y < 0$. $1 < -3$, which is false for all y. Case 1c: $y = 0$. $0 < 0$, which is false.

Therefore, $y > x > -3y$ and $y > 0$.

Case 2:

$$x + 3y < 0 \text{ and } x - y > 0$$

$y < x$

$$x > y$$

$$-3y > x > y$$

$$-3y > y$$

Similar to Cases 1a, 1b, and 1c, $y < 0$.

Therefore, $-3y > x > y$ and $y > 0$.

Since these are distinct cases, they are joined together using the union operator. Hence,

$$(y > 0 \text{ and } y > x > -3y) \text{ or } (y < 0 \text{ and} - 3y > x > y)$$

as required. □

Next, we will prove the converse. Since the hypothesis consists of two expressions joined by an "or" operator, both expressions must imply the conclusion.

Assuming the first expression:

$$y > 0 \text{ and } y > x > -3y$$

$$x > -3y$$

$$x + 3y > 0$$

$$y > x$$

$$x - y < 0$$

Since $x + 3y$ is positive and $x - y$ is negative, then $(x + 3y)(x - y)$ must be negative. That means $x^2 - xy + 3xy - 3y^2$ is negative, and $x^2 + 2xy - 3y^2$ is also negative, as required.

Assuming the second expression:

$$y < 0 \text{ and } -3y > x > y$$

$$x < -3y$$

$$x + 3y < 0$$

$$y < x$$

$$x - y > 0$$

Since $x + 3y$ is negative and $x - y$ is positive, then $(x + 3y)(x - y)$ must be negative. Similar to above, $x^2 + 2xy - 3y^2$ is also negative, as required.

Since we have proved the implication both ways, the if and only if statement is true. $\square$