MATH 135: Written Assignment 2

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Problem 1

1a)

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Since the truth values for all rows in columns $P \Rightarrow (Q \Rightarrow R)$ and $(P \wedge Q) \Rightarrow R$ are equivalent, therefore, these two expressions are logically equivalent.

1b)

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

Since the truth values in columns $P \Rightarrow (Q \Rightarrow R)$ and $(P \Rightarrow Q) \Rightarrow R$ differ in some rows, therefore, these two expressions are **not** logically equivalent.

Problem 2

2a)

$$\forall\, n \in \mathbb{N}, \exists\, t \in \mathbb{N}, (t > n) \wedge (s(n) = n)$$

2b)

$$\exists\, n \in \mathbb{N}, \forall\, t \in \mathbb{N}, (t \leq n) \vee (s(n) \neq n)$$

2c)

$$\forall\, n \in \mathbb{N}, (s(n) = n) \Rightarrow (\exists\, k \in \mathbb{N}, n = 2k)$$

2d)

$$\exists\, n \in \mathbb{N}, (s(n) = n) \wedge (\forall k \in \mathbb{N}, n \neq 2k)$$

Problem 3

3a) Proof:

$$\begin{aligned}
\text{LHS} &\equiv (\neg(P \Rightarrow Q)) \wedge R \\
&\equiv (\neg(\neg P \vee Q)) \wedge R && \text{expanding the "implies" operator} \\
&\equiv (\neg(\neg P) \wedge (\neg Q)) \wedge R && \text{De Morgan's Law} \\
&\equiv (P \wedge \neg Q) \wedge R && \text{double-negation identity} \\
&\equiv P \wedge (\neg Q \wedge R) && \text{associative property} \\
&\equiv P \wedge [\neg(Q \vee \neg R)] && \text{De Morgan's Law} \\
&\equiv P \wedge [\neg(Q \Rightarrow R)] && \text{reverting the "implies" operator} \\
&\equiv \text{RHS} && \blacksquare
\end{aligned}$$

3b) Proof:

$$\begin{aligned}
\text{RHS} &\equiv (\neg R \Rightarrow P) \wedge (Q \Rightarrow P) \\
&\equiv (R \vee P) \wedge (\neg Q \vee P) && \text{expanding the "implies" operator} \\
&\equiv (R \wedge \neg Q) \vee (R \wedge P) \vee (P \wedge \neg Q) \vee (P \wedge P) && \text{distributive property} \\
&\equiv (R \wedge \neg Q) \vee (P \wedge (R \vee \neg Q \vee \text{true})) \\
&\equiv P \vee (R \wedge \neg Q) \\
&\equiv P \vee \neg(Q \vee \neg R) && \text{De Morgan's Law} \\
&\equiv \text{LHS} && \blacksquare
\end{aligned}$$

Problem 4

4a)

$$\forall\, a, b, c, d \in \mathbb{N}, (\frac{a}{b} < \frac{c}{d}) \Rightarrow (\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d})$$

4b)

$$\exists\, a, b, c, d \in \mathbb{N}, (\frac{a}{b} < \frac{c}{d}) \wedge ((\frac{a}{b} > \frac{a+c}{b+d}) \vee (\frac{a+c}{b+d} > \frac{c}{d}))$$

4c)

$$\frac{a}{b} < \frac{c}{d}$$

4d)

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

4e) Let's prove that if $\frac{a}{b} < \frac{c}{d}$, then $(\frac{a}{b} < \frac{a+c}{b+d}$ and $\frac{a+c}{b+d} < \frac{c}{d})$.

$$\begin{aligned}
\frac{a}{b} &< \frac{c}{d} && \text{hypothesis assumption} \\
ad &< bc && \\
ad + ab &< bc + ab && \text{Add } ab \text{ to both sides} \\
a(b+d) &< b(a+c) && \text{Common factor } a \text{ and } b \\
\frac{a}{b} &< \frac{b+d}{a+c} && \blacksquare
\end{aligned}$$

$$\begin{aligned}
\frac{a}{b} &< \frac{c}{d} && \text{hypothesis assumption} \\
ad &< bc && \\
ad + cd &< bc + cd && \text{Add } cd \text{ to both sides} \\
d(a+c) &< c(b+d) && \text{Common factor } c \text{ and } d \\
\frac{a+c}{b+d} &< \frac{c}{d} && \blacksquare
\end{aligned}$$

Problem 5

5a) This statement is **false**. Counterexample $k = 0$:

$$\begin{aligned} &2^{(k^2)} + k^2 - 3k \\ &= 2^{(0^2)} + 0^2 - 3(0) \\ &= 2^0 + 0 - 0 \\ &= 1 \end{aligned}$$

Since 1 is not even, therefore, this universally quantified statement is false.

5b) This statement is **false**. We can prove that it's negation ($\forall\, r \in \mathbb{Z}, r^3 - 3r + 5$ is odd) is true. Since r^3 and $-3r$ both have the same parity as r, therefore, $r^3 - 3r$ is always even. An even number added to an odd number 5 is always odd. ■

5c) This statement is **true**. Proof:

let $x = 2^t$. x will always be positive due to the range of 2^t.

$$\begin{aligned} x + \frac{1}{x} &< \frac{65}{8} \\ 8x + \frac{8}{x} &< 65 \\ \frac{8x^2 + 8}{x} &< 65 \\ 8x^2 + 8 &< 65x \\ 8x^2 - 65x + 8 &< 0 \\ (8x - 1)(x - 8) &< 0 \\ \frac{1}{8} < 2^t &< 8 \\ -3 < t &< 3 \end{aligned} \qquad ■$$

5d) This statement is **true**. Example: $n = 41, a = 41, b = 41$

$$\begin{aligned} n^2 - n + 41 &= ab \\ 41^2 - 41 + 41 &= 41(41) \\ 1681 &= 1681 \end{aligned}$$